THE UNIVERSAL NOTATION.

Explanatory Statement of the Principles underlying the System of Universal Notation for Life Contingencies, adopted unanimously by the Second International Actuarial Congress on May 19, 1898. *Prepared by the Committee, (consisting of* M. Am. Bégault, *Brussels;* Dr. J. Karup, *Gotha;* Mr. George King, *London;* M. Léon Marie, *Paris; and* Dr. T. B. Sprague, *Edinburgh), appointed for that purpose.*

The Universal Notation, as embodied in the following Statement, has been taken, almost as it stands, from the "Key to the Notation" given in the *Institute of Actuaries Text-Book*, Part II, *Life Contingencies*. The simple principles on which the system is based are explained, and sufficient examples are given to make the application of these principles clear. Those portions of the Notation of the *Text-Book* have been omitted which are not of an International character, and, in accordance with a wish expressed by several at the Congress, emphasis has been laid on the advantages, in many cases, of using the symbols P, V, and W, in conjunction with other symbols, instead of alone; although, to meet the views strongly held by others, the option has been retained of employing these letters by themselves in simple cases, where confusion could not thereby be caused. Similarly, attention has been specially called to the use of accents affecting such symbols as P, V, &c., to denote office premiums, valuation premiums, &c., and the corresponding values of other functions.

The only changes that have been introduced into the Notation of the *Text-Book* are, to substitute the hitherto unappropriated letter W for (FP), to represent the Paid-up Policy, such a change having been urged by more than one speaker at the Congress; and to limit q, the probability of death, to a term of one year, Q being employed for longer terms.

The Notation for *Selection* is that which was officially adopted by the Institute of Actuaries of London for its two publications, *Select Life Tables* and *Joint Life Annuity Tables*.

Interest :—

i = the effective rate of interest, namely, the interest actually realised on 1 in one year.

$j_{(m)} = m\{(1+i)^{\frac{1}{m}}-1\}$ = the nominal rate of interest, convertible m times in a year, when the effective rate is i.

$i^{(m)}$ = the effective rate of interest, when interest is convertible m times in a year.

$\bar{\imath} = i^{(\infty)}$ = the effective rate of interest when interest is convertible momently.

$v = (1+i)^{-1}$ = the present value of 1 due a year hence.

$d = 1-v$ = the discount on 1 due a year hence.

$\delta = j_{(\infty)} = \log_e(1+i) = -\log_e(1-d)$ = the force of interest or the force of discount.

$a_{\overline{n}|} = v+v^2+v^3+\&c.+v^n$ = the value of an annuity for n years certain.

$s_{\overline{n}|} = 1+(1+i)+(1+i)^2+\&c.+(1+i)^{n-1}$ = the amount of an annuity for n years certain.

Mortality Tables :—

A letter enclosed in brackets, thus (x), denotes "a person aged x."

For each class of functions a principal letter is used, and its meaning in any particular case is defined by suffixes, &c., as circumstances may require.

We have

l = Number living.

d = Number dying.

L = Population.

p = Probability of living.

q, Q = Probability of dying, q being used for a term limited to one year, and Q for a longer term.

μ = Force of mortality.

m = Central death-rate.

e = Expectation of life.

E = Endowment.

a = Annuity, first payment to be made at the end of one period of payment.

$\mathbf{a}$ = Annuity, first payment to be made at once.

A = Assurance, or Single Premium.

V = Policy Value.

W = Paid-up Policy.*

P, π = Premium per annum, where P refers generally to net premiums, and π to special premiums, for instance, premiums for assurances with return of premiums.

The ages of the lives involved are denoted by letters placed as suffixes at the lower right corner of the principal symbol; and if two

* In the *Text-Book of the Institute of Actuaries* this is denoted by (FP).

or more letters in a suffix are not distinguished by any special mark, joint lives are intended. Thus,

l_x = Number living at age x according to the Mortality Table.
d_x = Number dying between the ages x and $x+1$.
p_x = Probability that (x) will live one year.
q_x = Probability that (x) will die within the year.
m_x = "Central death rate" for the year x to $x+1$, $=\mu_{x+\frac{1}{2}}$ approximately.
a_x = Annuity, first payment at the end of a year, to continue during the life of (x).
$\mathbf{a}_x$ = A similar annuity, first payment, however, to be made at once.
a_{xyz} = Annuity, first payment at the end of a year, to continue during the joint lives of (x), (y), and (z).
A_x = Assurance payable at the end of the year of the death of (x).
A_{xyz} = Assurance payable at the end of the year of the failure of the joint lives, (x), (y), and (z).

If one of the letters in the suffix is enclosed in a right angle, the symbol denotes a term-certain, and not the age of a life. Thus,

$a_{x\overline{n}|}$ = Annuity to continue during the joint duration of the life of (x) and a term of n years certain; that is, a temporary annuity for n years on the life of (x).
$\mathrm{A}_{x\overline{n}|}$ = Assurance payable at the end of the year of the death of (x) if he die within n years, or at the end of n years if (x) be then alive; that is, an endowment assurance for n years.

The suffix may consist only of a letter enclosed in a right angle, in which case a term-certain only is indicated. Thus,

$a_{\overline{n}|}$ = Annuity for n years certain.
$\mathrm{A}_{\overline{n}|}$ = v^n = Assurance payable at the end of n years certain.

If a perpendicular bar separates the letters in the suffix, then the status after the bar is to follow the status before the bar. Thus,

$a_{y|x}$ = Annuity on the life of (x) after the death of (y).
$\mathrm{A}_{z|xy}$ = Assurance payable on the failure of the joint lives (x) and (y) provided these lives both survive (z).

If a horizontal bar appears above the suffix, then survivors of the lives, and not joint lives, are intended. The number of survivors is denoted by a letter or number over the right end of the bar. If that letter, say r, is not distinguished by any mark, then the meaning is, *at least* r survivors; but if it is enclosed in square brackets, $[r]$, then the meaning is *exactly* r survivors. If no letter appears over the bar, then unity is supposed, and the meaning is *at least one* survivor. Thus,

$e_{\overline{xyz\ldots(m)}}^{\ r}$ = Expectation of life of the m lives and the last r survivors of them.
$p_{\overline{xyz\ldots(m)}}^{\ [r]}$ = Probability that exactly r lives out of m lives will survive a year.
$a_{\overline{xyz}}$ = Annuity on the last survivor of (x), (y), and (z).

When numerals are placed above or below the letters of the suffix, they designate the order in which the lives are to fail. The numeral placed *over* the suffix points out the life whose failure will finally determine the event; and the numerals placed *under* the suffix indicate the order in which the other lives involved are to fail. Thus,

$\left.\begin{matrix}{}^{1}Q_{xyz}\\ {}^{2}Q_{xyz}\\ {}^{3}Q_{xyz}\end{matrix}\right\}$ = Total probability that $\begin{cases}\text{first} & \text{of the three lives.}\\ \text{second} & \text{,,} \quad \text{,,}\\ \text{third} & \text{,,} \quad \text{,,}\end{cases}$

${}^{4}_{321}A_{wxyz}$ = Assurance payable at the end of the year of the death of (w) if he die last of the four lives, the other lives having failed in the order (z) first, (y) second, and (x) third.

$\left.\begin{matrix}{}^{\bar{2}|}_{1}a_{yz|x}\\ {}^{2}a_{yz|x}\end{matrix}\right\}$ or = $\begin{cases}\text{Annuity to } (x) \text{ after the failure of the survivor of } (y)\\ \text{and } (z), \text{ provided } (z) \text{ fail before } (y).\end{cases}$

${}^{z}_{3}A_{\overline{xy}:z}$ = Assurance payable at the end of the year of the death of the survivor of (x) and (y) if he die before (z).

When, for the sake of distinctness in the symbol, it is desired to separate the letters in the suffix, a colon is placed between them. A colon is used instead of a point or comma, so as not to confuse with decimals when numbers take the place of letters. Thus, we write $a_{x+n:y+n}$, and $A_{35:40}$.

A letter at the lower left corner of the principal symbol denotes the number of years involved in the probability or benefit in question. Thus,

${}_{n}p_{x}$ = Probability that (x) will live n years.
${}_{n}E_{x}$ = Value of endowment on (x), payable if he survive n years.

If the letter comes before a perpendicular bar, it shows that a deferred period is meant; while if the letter comes after a perpendicular bar, it shows that a temporary period is meant. Thus,

${}_{n|}q_{x}$ = Probability that (x) will die in a year, deferred n years; that is, that he will die in the $(n+1)$th year.
${}_{n}Q_{x}$ = Probability that (x) will die within n years.
${}_{n|}a_{x}$ = $a_{\bar{n}|x}$ = Annuity on (x) deferred n years; that is, the first payment to be made at the end of $(n+1)$ years.
${}_{|n}a_{x}$ = $a_{x\bar{n}|}$ = Temporary annuity on (x) for n years.
${}_{t|n}a_{x}$ = Intercepted annuity, that is, an annuity on (x), deferred t years, and after that to run for n years.

A letter in brackets at the *upper right* corner of the principal symbol shows the number of intervals into which the year is to be divided. Thus,

$a^{(m)}$ = Annuity payable by m instalments of $\frac{1}{m}$ each throughout the year.

$A_{x}^{(m)}$ = Assurance payable at the end of that fraction $\frac{1}{m}$ of a year in which (x) dies.

If the year be divided into an infinite number of infinitesimal parts, that is, if m become indefinitely great, then, instead of writing (∞), a bar is placed over the principal symbol. Thus,

$\bar{a}$ = Continuous or momently annuity.
$\bar{\mathrm{A}}$ = Assurance payable at the moment of death.

A small circle placed over the principal symbol shows that the benefit is to be complete. Thus,

$\mathring{e}$ = Complete expectation of life.
$\mathring{a}$ = Complete annuity.

In the case of Reversionary Annuities, distinction has sometimes to be made between those where the times of year at which payments are to take place, are determined at the outset; and those where the times depend on the moment of failure of the preceding status. Thus,

$a_{y|x}$ = Annuity to (x), first payment at the end of the year of the death of (y), or, on the average, six months after his death.
$\hat{a}_{y|x}$ = Annuity to (x), first payment one year after the death of (y).
$\mathring{\hat{a}}_{y|x}$ = Complete annuity to (x), first payment one year after the death of (y).

The symbols P for Premium, V for Policy-Value, and W for Free or Paid-up Policy, are usually to be employed in conjunction with the other symbol denoting the benefit. Thus,

$\mathrm{P}(\mathrm{A}^1_{xy})$ = The annual premium for a contingent assurance on (x) against (y).
${}_t\mathrm{V}(\mathrm{A}_{x\bar{n}|})$ = Value after t years of an endowment assurance on (x).
${}_t\mathrm{W}(\mathrm{A}_x)$ = Paid-up Policy after t years which is the equivalent of an ordinary policy on (x) that has been t years in force.

Suffixes, &c., showing the conditions of the benefit, are to be attached to the principal letter, and suffixes, &c., showing the conditions of payment of the premium, are to be attached to the subsidiary symbol P. Thus,

${}_n\mathrm{P}(\bar{\mathrm{A}}_x)$ = Annual premium, payable for n years only, for an assurance payable at the moment of the death of (x).
$\mathrm{P}_{xy}(\mathrm{A}_x)$ = Annual premium, payable during the joint lives of (x) and (y), for an assurance payable at the end of the year of the death of (x).
${}_t\mathrm{P}^{(m)}(\mathrm{A}_{x\bar{n}|})$ = Premium per annum for t years only, payable by m instalments throughout the year, for an endowment assurance for n years on (x).

It is permissible, however, in simple cases where no confusion can arise, to employ the letters P, V, and W as symbols by themselves. Thus, we may write P^1_{xy} for $\mathrm{P}(\mathrm{A}^1_{xy})$; ${}_t\mathrm{V}_{x\bar{n}|}$ for ${}_t\mathrm{V}(\mathrm{A}_{x\bar{n}|})$; and ${}_t\mathrm{W}_x$ for ${}_t\mathrm{W}(\mathrm{A}_x)$.

In particular investigations, where modified values of functions are in question, such modification may be denoted by adding accents

to the symbols. Thus, where a premium other than the net premium (a valuation premium) is used in a valuation, it may be denoted by P′; and the corresponding policy-value may be denoted by V′. Similarly, the "office" or "commercial" premium may be denoted by P″, and the surrender-value of a policy by V″, and the "office" paid-up policy by W″.

The following compound symbols are used:

(Ia) = Annuity { commencing at 1, and
(IA) = Assurance { increasing 1 per annum.
(va) = Varying annuity.
(vA) = Varying assurance.

If the whole benefit is to be temporary, the symbol of limitation is placed outside the brackets. Thus.

$(Ia)_{x\overline{n}|}$ = Temporary increasing annuity.
$(IA)^{1}_{x\overline{n}|}$ = „ „ assurance.
$(va)_{x\overline{n}|}$ = „ varying annuity.
$(vA)^{1}_{x\overline{n}|}$ = „ „ assurance.

If only the increase or the variation is to be temporary, but the benefit to be for the whole of life, then the symbol of limitation is placed immediately after the symbol I or v. Thus,

$(I_{\overline{n}|}a)_x$ = Whole-life annuity
$(I_{\overline{n}|}A)_x$ = „ assurance } increasing for n years.
$(v_{\overline{n}|}a)_x$ = „ annuity
$(v_{\overline{n}|}A)_x$ = „ assurance } varying for n years.

Commutation Columns:—

$$\begin{aligned} D_x &= v^x l_x. \\ N_x &= D_{x+1}+D_{x+2}+D_{x+3}+\&c. \\ \mathbb{N}_x \text{ or } {}_{x}\} &= D_x+D_{x+1}+D_{x+2}+\&c. \\ S_x &= N_x+N_{x+1}+N_{x+2}+\&c. \\ C_x &= v^{x+1}d_x. \\ M_x &= C_x+C_{x+1}+C_{x+2}+\&c. \\ R_x &= M_x+M_{x+1}+M_{x+2}+\&c. \end{aligned}$$

When it is desired to construct the assurance columns so as to give directly assurances payable at the moment of death, the symbols are distinguished by a bar placed over them. Thus,

$$\begin{aligned} \overline{C}_x &= v^{x+\frac{1}{2}}d_x, \text{ approximately.} \\ \overline{M}_x &= \overline{C}_x+\overline{C}_{x+1}+\overline{C}_{x+2}+\&c. \\ \overline{R}_x &= \overline{M}_x+\overline{M}_{x+1}+\overline{M}_{x+2}+\&c. \end{aligned}$$

Joint Lives:—

$$\begin{aligned} D_{xy} &= v^{\frac{x+y}{2}} l_x l_y \\ N_{xy} &= D_{x+1:y+1}+D_{x+2:y+2}+D_{x+3:y+3}+\&c. \\ C_{xy} &= v^{\frac{x+y}{2}+1}(l_x l_y - l_{x+1}l_{y+1}) \\ M_{xy} &= C_{xy}+C_{x+1:y+1}+C_{x+2:y+2}+\&c. \\ C^{1}_{xy} &= v^{\frac{x+y}{2}+1} d_x l_{y+\frac{1}{2}} \\ M^{1}_{xy} &= C^{1}_{xy}+C^{\frac{1}{2}}_{x+1:y+1}+C^{\frac{1}{2}}_{x+2:y+2}+\&c. \end{aligned}$$

Selection :—

Square brackets in the suffix to a symbol denote the age at which the life was selected. Any additional term in the suffix, not enclosed in square brackets, denotes the number of years which have elapsed since selection. The total suffix, therefore, denotes, as usual, the present age of the life. Thus,

$a_{[x]}$ = value of an annuity on a life now aged x and now select.
$a_{[x]+n}$ = value of an annuity on a life now aged $x+n$, and select n years ago at age x.
$a_{[x-n]+n}$ = value of an annuity on a life now aged x, and select n years ago at age $x-n$.

Similarly for other functions.

Notes :—

Dr. Sprague, while approving in general of the list of Symbols and Functions tabulated above, and therefore accepting this Schedule, thinks that the arrangement and explanations might be improved, and that some additional Symbols might, with advantage, be included. He therefore proposes to prepare a Paper on the subject, and submit it to the Paris Congress of 1900.

Dr. Sprague and Mr. King consider that P, V, and W, should be used alone, and not in combination with other Symbols (such as A or a), except in the case of complicated benefits where confusion might otherwise arise.

Dr. Karup desires to point out that he considers the Symbols $i^{(m)}$ and $\bar{\imath}$ to be superfluous, as i, $j_{(m)}$ and δ sufficed to express all the relations between the effective and the nominal rates of interest. For from the definition of $j_{(m)}$ it follows that $i=\left(1+\frac{j_{(m)}}{m}\right)^m-1$, and therefore that $i^{(m)}$ and i are in fact identical. And similarly it follows, from the definition of δ, that $i=e^{\delta}-1=e^{j_{(m)}}-1$, so that $\bar{\imath}$ and i are also identical.